

# Large cycles in 4-connected graphs

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## Abstract

Every 4-connected graph  $G$  with minimum degree  $\delta$  and connectivity  $\kappa$  either contains a cycle of length at least  $4\delta - \kappa - 4$  or every longest cycle in  $G$  is a dominating cycle.

We consider only finite undirected graphs without loops or multiple edges. Let  $n$  denote the order,  $\delta$  the minimum degree,  $\kappa$  the connectivity and  $c$  the circumference (the length of a longest cycle) of a graph  $G$ . A cycle  $C$  is a Hamilton cycle if  $|C| = n$  and is a dominating cycle if every edge of  $G$  has a vertex in common with  $C$ . A cycle  $C$  is said to be a  $D_3$ -cycle if every path of length at least 2 has a vertex in common with  $C$ .

In 2008, Yamashita [3] obtained a degree sum condition for dominating cycles which yields the following.

**Theorem A** [3]. Let  $G$  be a 3-connected graph. If  $\delta \geq (n + \kappa + 3)/4$ , then any longest cycle in  $G$  is a dominating cycle.

In this paper we prove, in fact, the reverse version of Theorem A.

**Theorem 1.** Let  $G$  be a 4-connected graph. Then either  $c \geq 4\delta - \kappa - 4$  or every longest cycle in  $G$  is a dominating cycle.

In order to prove Theorem 1, we need the following result due to Jung [2].

**Theorem B** [2]. Let  $G$  be a 4-connected graph. Then either  $c \geq 4\delta - 8$  or every longest cycle in  $G$  is a  $D_3$ -cycle.

A good reference for any undefined terms is [1]. The set of vertices of a graph  $G$  is denoted by  $V(G)$  and the set of edges by  $E(G)$ . For  $S$  a subset of  $V(G)$ , we denote by  $G \setminus S$  the maximum subgraph of  $G$  with vertex set  $V(G) \setminus S$ . For a subgraph  $H$  of  $G$  we use  $G \setminus H$  short for  $G \setminus V(H)$ . We denote by  $N(x)$  the neighborhood of a vertex  $x$  in a graph  $G$  with  $d(x) = |N(x)|$ . Furthermore, for a subgraph  $H$  of  $G$  and  $X \subseteq V(G)$ , we define  $N_H(X) = N(X) \cap V(H)$ .

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Paths and cycles in a graph  $G$  are considered as subgraphs of  $G$ . If  $Q$  is a path or a cycle, then the length of  $Q$ , denoted by  $|Q|$ , is  $|E(Q)|$ . We write a cycle  $C$  with a given orientation by  $\vec{C}$ . For  $x, y \in V(C)$ , we denote by  $x\vec{C}y$ , or sometimes by  $C[x, y]$ , the subpath of  $C$  in the chosen direction from  $x$  to  $y$ . For  $C[x^+, y^+]$  we also write  $C(x, y)$ . For  $x \in V(C)$ , we denote the  $h$ -th successor and the  $h$ -th predecessor of  $x$  on  $\vec{C}$  by  $x^{+h}$  and  $x^{-h}$ , respectively. We abbreviate  $x^{+1}$  and  $x^{-1}$  by  $x^+$  and  $x^-$ , respectively. For  $X \subset V(C)$ , we define  $X^{+h} = \{x^{+h} | x \in X\}$  and  $X^{-h} = \{x^{-h} | x \in X\}$ .

Henceforth, we use the following notation. Let  $G$  be a 4-connected graph and  $C$  be a  $D_3$ -cycle in  $G$  with  $x_1x_2 \in E(G \setminus C)$ . We denote

$$R = N_C(x_1) \cup N_C(x_2), \quad M = N_C(x_1) \cap N_C(x_2),$$

$$A = R \setminus M, \quad A_1 = N_C(x_1) \setminus M, \quad A_2 = N_C(x_2) \setminus M,$$

$$Y = R \cup R^+ \cup M^{+2}.$$

**Lemma 1.**  $|Y| \geq 2d(x_1) + 2d(x_2) - |M| - 4 \geq 4\delta - |M| - 4$ .

**Proof.** Since  $C$  is extreme,  $R, R^+$  and  $M^{+2}$  are pairwise disjoint. Observing that  $R = A \cup M$ , we get

$$|Y| = |R| + |R^+| + |M^{+2}| = 2|R| + |M| = 3|M| + 2|A|.$$

Next, since  $|A_i| = d(x_i) - |M| - 1$  and  $|A| = |A_1| + |A_2|$ ,

$$|Y| \geq 2d(x_1) + 2d(x_2) - |M| - 4 \geq 4\delta - |M| - 4. \quad \Delta$$

**Lemma 2.** Let  $P$  be a longest  $(x, y)$ -path, having only  $y$  in common with  $C$ . If  $y \in R^+$ , then either  $|P| = 0$  or  $c \geq 4\delta - \kappa - 3$ , and if  $y \in M^{+2}$ , then either  $|P| \leq 1$  or  $c \geq 4\delta - \kappa - 3$ .

**Proof.** Let  $\xi_1, \dots, \xi_t$  be the elements of  $R$ , occuring on  $\vec{C}$  in a consecutive order. Assume w.l.o.g. that  $y \in \{\xi_1^+, \xi_1^{+2}\}$  and choose  $w \in N_C(x) \setminus \{y\}$ . Let  $w \in V(\xi_i^+ \vec{C} \xi_{i+1})$  for some  $i \in \{1, \dots, t\}$ . Let  $Q$  be a longest path connecting  $\xi_i$  to  $\xi_1$  and passing through  $\{x_1, x_2\}$ . Put  $C' = \xi_i Q \xi_1 \vec{C} w x P y \vec{C} \xi_i$ .

**Case 1.**  $y \in R^+$ .

Assume that  $|P| \geq 1$ . Since  $C$  is a  $D_3$ -cycle,  $1 \leq |P| \leq 2$ .

**Case 1.1.**  $P = xy$ .

Since  $|C'| \leq |C|$ , we have  $|\xi_i \vec{C} w| \geq 4$  if  $\xi_i \in M$  and  $|\xi_i \vec{C} w| \geq 3$  if  $\xi_i \in A$ . It means that

$$(N^-(x) \setminus \{y^-, y^+\}) \cap Y = \emptyset.$$

Observing also that  $d(x) \geq |M| + 1$  and using Lemma 1, we get

$$c \geq |Y| + |N^-(x) \setminus \{y^-, y^+\}| \geq 4\delta + d(x) - |M| - 6 \geq 4\delta - \kappa - 3.$$

**Case 1.2.**  $P = xzy$  for some vertex  $z$ .

Observing that  $(N_C^-(x) \setminus \{y^-\}) \cap Y = \emptyset$  and  $|N_C(x)| \geq |N(x)| - 1$ , we can argue as in Case 1.1.

**Case 2.**  $y \in M^{+2}$ .

Assume that  $|P| \geq 2$ . Since  $C$  is a  $D_3$ -cycle, we have  $|P| = 2$ , i.e.  $P = xzy$  for some  $z \in V(G)$ . Due to  $|P| = 2$ , we can obtain  $(N_C^-(x) \setminus \{y^-\}) \cap Y = \emptyset$  and further we can argue as in Case 1.1.  $\Delta$

**Lemma 3.** Let  $S$  be a minimum cut-set of  $G$ . Then either  $c \geq 4\delta - \kappa - 3$  or  $\{x_1, x_2\} \cap S = \emptyset$  for each  $x_1x_2 \in E(G \setminus C)$ .

**Proof.** Choose a longest cycle  $C$  such that  $|V(C) \cap S|$  is as great as possible and let  $x_1x_2 \in E(G \setminus C)$  with  $\{x_1, x_2\} \cap S \neq \emptyset$ . Let  $\xi_1, \dots, \xi_t$  be the elements of  $R$ , occuring on  $\vec{C}$  in a consecutive order. Since  $C$  is extreme,  $(R^+ \cup M^{+2}) \cap R = \emptyset$ . Further, since  $|V(C) \cap S|$  is maximum,  $M_1^{+3} \cap R = \emptyset$ . Observing also that  $|M_2| \leq \kappa - 1$  and using Lemma 1, we get

$$c \geq |Y| + |M_1^{+3}| \geq 4\delta - |M| - 4 + |M_1| = 4\delta - |M_2| - 4 \geq 4\delta - \kappa - 3. \quad \Delta$$

**Proof of Theorem 1.** Let  $G$  be a 4-connected graph,  $S$  be a minimum cut-set in  $G$  and  $H_1, \dots, H_h$  be the components of  $G \setminus S$ . If  $c \geq 4\delta - 8$ , then we are done, since  $4\delta - 8 \geq 4\delta - \kappa - 4$ . Otherwise, by Theorem B, every longest cycle in  $G$  is a  $D_3$ -cycle. Let  $C$  be any longest cycle and  $x_1x_2 \in E(G \setminus C)$ . Assume w.l.o.g. that  $x_1x_2 \in V(H_1)$ . By Lemma 3,  $\{x_1, x_2\} \cap S = \emptyset$ . Abbreviate,  $V_1 = V(H_1) \cup S$ . Assume first that  $Y \subseteq V_1$ . By Lemma 1,

$$|V(C) \cap V_1| \geq |Y| \geq (2d(x_1) + d(x_2)) + d(x_2) - |M| - 4 \geq 3\delta - 3.$$

If  $V(H_2) \subseteq V(C)$ , then  $|V(C \cap H_2)| \geq \delta - \kappa + 1$  and

$$c \geq |V(C \cap V_1)| + |V(C \cap H_2)| \geq 4\delta - \kappa - 2.$$

Otherwise, we choose  $y \in V(H_2 \setminus C)$ . Since  $|N_C(y)| \geq \delta - 1$ , we have  $|V(C \cap H_2)| \geq |N_C(y)| - |S|$  and

$$c \geq |V(C \cap V_1)| + |V(C \cap H_2)| \geq 4\delta - \kappa - 4.$$

Now let  $Y \not\subseteq V_1$ . Since  $R \subseteq V_1$ , we have  $R^+ \cup M^{+2} \not\subseteq V_1$ .

**Case 1.**  $R^+ \cap V(H_2) \neq \emptyset$ .

Let  $y \in R^+ \cap V(H_2)$ . By Lemma 2,  $N(y) \subseteq V(C)$  and by standard arguments,  $N(y) \cap (R^+ \cap M^{+2}) = \emptyset$ . Since  $|N(y) \cap Y| = |N(y) \cap R| \leq \kappa$ , we have by Lemma 1,

$$\begin{aligned} c &\geq |Y| + |N(y)| - |N(y) \cap Y| \geq |Y| + \delta - \kappa \\ &\geq (2d(x_1) + d(x_2) + \delta) + (d(x_2) - |M|) - \kappa - 4 \geq 4\delta - \kappa - 3. \end{aligned}$$

**Case 2.**  $R^+ \cup V(H_2) = \emptyset$ .

We have  $R \cup R^+ \subseteq V_1$  and  $M^{+2} \cap V(H_2) \neq \emptyset$ . Then it is easy to see that for each  $y \in M^{+2} \cap V(H_2)$ ,  $|N(y) \cap Y| \leq \kappa$ . If  $N(y) \subseteq V(C)$ , then  $c \geq |Y| + |N(y) \setminus Y| \geq |Y| + |N(y)| - \kappa$  and we can argue as in Case 1. Let  $N(y) \not\subseteq V(C)$  and  $z \in N(y) \setminus V(C)$ . By Lemma 2,  $N(z) \subseteq V(C)$ . If  $z \in V(H_2)$ , then by standard arguments,  $N(z) \cap (R^+ \cup M^{+2}) = \emptyset$ . Hence

$$c \geq |Y| + |N(y) \setminus Y| = |Y| + |N(y)| - |N(y) \cap R| \geq |Y| + |N(y)| - \kappa$$

and we can argue as in Case 1. Let  $z \notin V(H_2)$ . Then we can assume that  $N(y) \setminus V(C) \subseteq S$ . Set  $D = N(y) \cap (S \setminus V(C))$ . Since  $|N(y) \cap Y| \leq \kappa - |D|$  and  $|N(y) \cap V(C)| \geq \delta - |D|$ , we have

$$\begin{aligned} c &\geq |Y| + |(N(y) \cap V(C)) \setminus Y| \\ &= |Y| + |N(y) \cap V(C)| - |N(y) \cap Y| \geq |Y| + \delta - \kappa \end{aligned}$$

and again we can argue as in Case 1.  $\Delta$

## References

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